10. Nekonečná geometrická řada

Příklady

① Dána geometrická posloupnost $(a_n)_{n=1}^{\infty}$, $a_n = (\frac{1}{2})^n$:

a) Určete prvních 6 členů, kvocient a limitu

$$a_n = \left(\tfrac{1}{2}\right)^n = \tfrac{1}{2}\cdot\left(\tfrac{1}{2}\right)^{n-1}$$

GP: $a_n = \frac{1}{2}\cdot(\frac{1}{2})^{n-1} \Rightarrow a_1 = \frac{1}{2},\ q = \frac{1}{2}$

$a_n = a_1 q^{n-1}$

$|q|<1 \Rightarrow \lim\limits_{n\to\infty} (\frac{1}{2})^n = 0$

podmínka konvergence

členy: $a_1 = \frac{1}{2}$ $a_2 = \frac{1}{4}$ $a_3 = \frac{1}{8}$ $a_4 = \frac{1}{16}$ $a_5 = \frac{1}{32}$ $a_6 = \frac{1}{64}$

b) vytvořte novou posloupnost $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$ a zjistěte, zda je konvergentní.

$s_1 = a_1 = \frac{1}{2}$

$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$\vdots$

$$s_n = \underbrace{a_1 \frac{q^n - 1}{q-1}}_{\text{součet } n \text{ členů GP}} = a_1 + a_2 + \ldots + a_n = \frac{1}{2}\,\frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \frac{1}{2}\,\frac{(\frac{1}{2})^n - 1}{-\frac{1}{2}} = 1 - \left(\tfrac{1}{2}\right)^n$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \left(\tfrac{1}{2}\right)^n\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^n = 1 - 0 = \underline{1}$$

VĚTA

- je-li $(a_n)_{n=1}^{\infty}$ geometrická posloupnost $[a_n = a_1 q^{n-1}]$ s kvocientem $|q|<1$, potom posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$, je také konvergentní a

$$\lim_{n\to\infty} s_n = \frac{a_1}{1-q}$$

- je-li $|q| \geq 1$, potom $(s_n)_{n=1}^{\infty}$ není konvergentní

[důkaz:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_1 \frac{q^n - 1}{q-1} = \lim_{n\to\infty} \underbrace{\frac{a_1}{q-1}}_{\in R} \cdot \lim_{n\to\infty}(q^n - 1) =$$

$$= \lim_{n\to\infty} \underbrace{\frac{a_1}{q-1}}_{\in R} \left(\underbrace{\lim_{n\to\infty} q^n}_{=0 \text{ pro } |q|<1} - \lim_{n\to\infty} 1\right) = \frac{a_1}{q-1}(0-1) = -\frac{a_1}{q-1} = \frac{a_1}{1-q} \quad \text{cbd.}$$

]

② Dána geometrická posloupnost $((-1)^n)_{n=1}^{\infty}$. Určete několik členů posloupnosti $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$ a dále zjistěte, zda je tato posloupnost konvergentní.

GP: $a_1 = -1 \quad a_2 = 1 \quad a_3 = -1 \quad a_4 = 1 \ \ldots \quad q = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = -1$

$|q| < 1$ neplatí
podmínka konv. GP nepl. $\Big\} \Rightarrow ((-1)^n)_{n=1}^{\infty}$ je divergentní

$\Rightarrow (s_n)_{n=1}^{\infty}$ je také divergentní

$$(s_n)_{n=1}^{\infty}: \left.\begin{array}{l} s_1 = a_1 = -1 \\ s_2 = a_1 + a_2 = -1+1 = 0 \\ s_3 = a_1 + a_2 + a_3 = -1+1-1 = -1 \\ s_4 = -1+1-1+1 = 0 \\ \vdots \end{array}\right\} \Rightarrow \begin{array}{l} s_{2k-1} = -1 \\ s_{2k} = 0 \end{array}$$

$\Big\} \Rightarrow$ posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$ je divergentní

③ Dána posloupnost $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n(n+1)}$. Určete, pokud existuje, $\lim\limits_{n\to\infty} s_n$, kde $s_n = a_1 + a_2 + \ldots + a_n$

$a_1 = \frac{1}{2},\ a_2 = \frac{1}{6},\ a_3 = \frac{1}{12},\ a_4 = \frac{1}{20}, \ldots$ není GP $\left[\frac{a_2}{a_1} \neq \frac{a_3}{a_2} \neq q \ldots\right]$, musíme určit vzorec pro s_n

$$(s_n)_{n=1}^{\infty}: \left.\begin{array}{l} s_1 = \frac{1}{2} \\ s_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\ s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\ s_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \\ \vdots \end{array}\right\} \begin{array}{l} ?\ s_n = \frac{n}{n+1} \quad \text{(dokázat matem. indukcí)} \\ ?\ \lim\limits_{n\to\infty} s_n = 1 \end{array}$$

$s_n = \frac{n}{n+1}$ (předp., než jsme dok. platnost mat. ind.)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n \cdot 1}{n(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = \underset{\sim}{1}$$

DEFINICE

Nechť je dána posloupnost $(a_n)_{n=1}^{\infty}$.
NEKONEČNOU ŘADOU nazýváme výraz $a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum\limits_{n=1}^{\infty} a_n$.
Členy nekonečné řady jsou členy dané posloupnosti.

- NEKONEČNÁ ŘADA je KONVERGENTNÍ $\Leftrightarrow$ posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$, je konvergentní a

součet této řady je $\quad s = \lim\limits_{n\to\infty} s_n = \sum\limits_{n=1}^{\infty} a_n \quad s = \lim\limits_{n\to\infty} s_n = \frac{a_1}{1-q}$

př. $(2n)_{n=1}^{\infty}$

$\sum\limits_{n=1}^{\infty} 2n = 2+4+6+8+\ldots$ divergentní

$\left(\left(-\frac{1}{3}\right)^{n-1}\right)_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \overset{a_1}{1} - \overset{a_2}{\frac{1}{3}} + \overset{a_3}{\frac{1}{9}} - \frac{1}{27} + \ldots = \frac{1}{1-(-\frac{1}{3})} = \frac{1}{\frac{4}{3}} = \underset{\sim}{\frac{3}{4}}$$

$a_1 = 1 \quad q = \frac{a_2}{a_1} = \frac{-\frac{1}{3}}{1} = -\frac{1}{3} \quad |q| < 1 \Rightarrow$ podm. konv. pl.

DEFINICE

je-li $(a_n)_{n=1}^{\infty}$ geometrická posloupnost s kvocientem q, $a_1 \neq 0$, pak příslušnou nekonečnou řadu

$$\sum_{n=1}^{\infty} a_n = a_1 + a_1q + a_1q^2 + a_1q^3 + \dots + a_1q^{n-1} + \dots$$

nazýváme NEKONEČNÁ GEOMETRICKÁ ŘADA

VĚTA

je-li $(a_n)_{n=1}^{\infty}$ GEOMETRICKÁ POSLOUPNOST s kvocientem q, pak NEKONEČNÁ GEOMETRICKÁ ŘADA je pro $|q|<1$ $(a_1 \neq 0)$ KONVERGENTNÍ a její součet je
$$s = \lim_{n\to\infty} s_n = \frac{a_1}{1-q} = \sum_{n=1}^{\infty} a_n$$

[pro $|q| \geq 1$ je divergentní]

př. $\left(\left(-\frac{1}{3}\right)^{n-1}\right)_{n=1}^{\infty}$ $\quad$ [$\left(-\frac{1}{3}\right)^{1-1} = \left(-\frac{1}{3}\right)^{0} = 1 = a_1$, $a_2 = \left(-\frac{1}{3}\right)^{2-1} = -\frac{1}{3}$, $a_3 = \left(-\frac{1}{3}\right)^{2} = \frac{1}{9}$, $a_4 = -\frac{1}{27}$

$\Rightarrow$ GP: $q = -\frac{1}{3}$ $\quad |q|<1$! podm. konv. platí $\Rightarrow$

$\Rightarrow$ konv. i GŘ a $s = \lim_{n\to\infty} s_n = \frac{a_1}{1-q}$]

$$s = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

[$s = \lim_{n\to\infty} s_n = \frac{a_1}{1-q}$ $\quad$ pro GŘ s $|q|<1$, $a_1 \neq 0$]

4) Zjistěte, zda jsou řady konvergentní, příp. určete jejich součet

a) $\sum_{n=1}^{\infty} \frac{2n-1}{2} = \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \dots$ $\quad$ divergentní

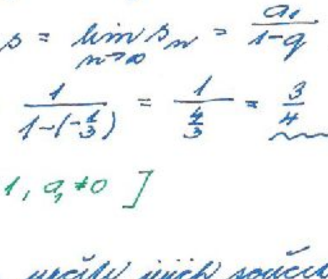
[$\lim_{n\to\infty} \frac{2n-1}{2} = +\infty$ $\quad$ NEVLASTNÍ LIMITA $(+\infty, -\infty) \Rightarrow$ divergentní]

b) $\sum_{n=1}^{\infty} \frac{2n-1}{n} = \frac{1}{1} + \frac{3}{2} + \frac{5}{3} + \frac{7}{4} + \frac{9}{5} + \dots$ $\quad$ konvergentní

$$\lim_{n\to\infty} \frac{2n-1}{n} = \lim_{n\to\infty} \frac{n\left(2-\frac{1}{n}\right)}{n} = \lim_{n\to\infty} \left(2 - \frac{1}{n}\right) = 2 - 0 = 2$$

posloupnost $\left(\frac{2n-1}{n}\right)_{n=1}^{\infty}$ konvergentní

(musíme pro s_n ověřit) $\Rightarrow$ z grafu



$s_1 = 1$

$s_2 = 1 + \frac{3}{2} = \frac{5}{2} = 2{,}5$

$s_3 = 1 + \frac{3}{2} + \frac{5}{3} = \frac{6+9+10}{6} = \frac{25}{6} = 4\frac{1}{6}$

$s_4 = s_3 + a_4 = \frac{25}{6} + \frac{7}{4} = \frac{50+21}{12} = \frac{71}{12} = 5\frac{11}{12}$

$(s_n)_{n=1}^{\infty}$ divergentní (z grafu)

c) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{5}}{2^{n-1}} = \frac{\sqrt{5}}{2^0} + \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2^2} + \frac{\sqrt{5}}{2^3} + \dots = \sqrt{5} + \sqrt{5}\cdot 2^{-1} + \sqrt{5}\cdot 2^{-2} + \sqrt{5}\cdot 2^{-3} + \dots$

(1. zp.) GŘ: $q = \frac{\frac{\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{\sqrt{5}}{2^2}}{\frac{\sqrt{5}}{2}} = \frac{1}{2}$ podm. konv. $|q|<1$ platí

$\Rightarrow s = \lim\limits_{n\to\infty} s_n = \frac{a_1}{1-q} = \frac{\sqrt{5}}{1-\frac{1}{2}} = \frac{\sqrt{5}}{\frac{1}{2}} = 2\sqrt{5}$

(2. zp.) upravíme

$\sum\limits_{n=1}^{\infty} \frac{\sqrt{5}}{2^{n-1}} = \sqrt{5} + \sqrt{5}\cdot 2^{-1} + \sqrt{5}\cdot 2^{-2} + \dots$

$= \sqrt{5}\left(1 + 2^{-1} + 2^{-2} + \dots\right) = \sqrt{5}\Big(\underbrace{\left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \dots}_{\text{GŘ: } a_1 = 1\ [(\frac{1}{2})^0]}\Big)$

$q = \frac{1}{2}$ $|q|<1$ pl.

$\Rightarrow s = \sqrt{5}\, s' = \sqrt{5}\cdot 2 = 2\sqrt{5}$ $\qquad s' = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

d) $\sum\limits_{n=1}^{\infty} \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{n-1} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^0 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^1 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 + \dots =$

$= 1 + \frac{\sqrt{3}-1}{\sqrt{2}} + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 + \dots$

GŘ: $a_1 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^0 = 1$

$q = \frac{\sqrt{3}-1}{\sqrt{2}} \doteq \frac{1{,}7-1}{1{,}4} = \frac{0{,}7}{1{,}4} \doteq \frac{1}{2}$ podm. konv. $|q|<1$ platí

$\Rightarrow s = \frac{a_1}{1-q} = \frac{1}{1-\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{1}{\frac{\sqrt{2}-\sqrt{3}+1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-\sqrt{3}+1}$

e) $\sum\limits_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n = \sum\limits_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \underbrace{\left(-\frac{2}{3}\right)^1 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3 + \dots}$

GŘ: $q = -\frac{2}{3}$ $a_1 = -\frac{2}{3}$

$|q|<1$ platí podm. konv.

$\Rightarrow s = \frac{a_1}{1-q} = \frac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{-\frac{2}{3}}{\frac{5}{3}} = -\frac{6}{15} = -\frac{2}{5}$

⑤ Zjistěte, zda je nekonečná řada konvergentní a určete její součet

a) $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \dots = 1 \underbrace{+ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}_{\text{GŘ}}$

$[1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ GP: $a_1 = 1$ $q = \frac{1}{2} \Rightarrow |q|<1$ konv.$]$

GŘ: $a_1 = 1$, $q = \frac{1}{2}$

$|q|<1$ podm. konv. platí $\Rightarrow$

$\Rightarrow s = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

b) $\sum_{n=1}^{\infty} 10^{-n} = 10^{-1} + 10^{-2} + 10^{-3} = \underbrace{\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots} = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

GŘ: $q = \frac{a_2}{a_1} = \frac{\frac{1}{100}}{\frac{1}{10}} = \frac{10}{100} = \frac{1}{10}$

$a_1 = \frac{1}{10}$ $\quad |q| < 1$ podmínka konvergence GŘ platí =>

$\Rightarrow s = \frac{a_1}{1-q} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{10}{90} = \frac{1}{9}$

tedy: $s = 0{,}1 + 0{,}01 + 0{,}001 + \ldots = 0{,}1111\ldots = 0{,}\overline{1}$

$\Big\} \Rightarrow 0{,}\overline{1} = \frac{1}{9}$

⑥ napište ve tvaru zlomku s celočíselným čitatelem a jmenovatelem

a) $0{,}\overline{4} = 0{,}4444\ldots = 0{,}4 + 0{,}04 + 0{,}004 + 0{,}0004 + \ldots =$

1. zp. $\quad = \underbrace{4 \cdot 10^{-1} + 4 \cdot 10^{-2} + 4 \cdot 10^{-3} + 4 \cdot 10^{-4} + \ldots}_{GŘ}$

$q = \frac{4 \cdot 10^{-2}}{4 \cdot 10^{-1}} = 10^{-1} = \frac{1}{10}$ $\qquad s = \frac{a_1}{1-q} = \frac{\frac{4}{10}}{1-\frac{1}{10}} = \frac{\frac{4}{10}}{\frac{9}{10}} = \frac{40}{90} = \frac{4}{9}$

$a_1 = 4 \cdot 10^{-1}$ $\quad |q| < 1$ platí

$0{,}\overline{4} = \frac{4}{9}$

lze i s využ. př. ⑤ b)

2. zp. $\quad 0{,}\overline{4} = 0{,}4444\ldots = 0{,}4 + 0{,}04 + 0{,}004 + 0{,}0004 + \ldots =$

$= 4(\underbrace{0{,}1 + 0{,}01 + 0{,}001 + 0{,}0001 + \ldots}_{GŘ}) =$

$q = \frac{0{,}01}{0{,}1} = 0{,}1 = \frac{1}{10}$

$0{,}\overline{4} = 4s = 4 \cdot \frac{1}{9}$ $\qquad a_1 = 0{,}1 \quad s = \frac{a_1}{1-q} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$

$0{,}\overline{4} = \frac{4}{9}$

[v 1. roč. jsme řešili i jinak - viz dále]

b) $-2{,}\overline{5} = -(2 + 0{,}5 + 0{,}05 + 0{,}005 + \ldots) = -2 - (\underbrace{5 \cdot 10^{-1} + 5 \cdot 10^{-2} + 5 \cdot 10^{-3} + \ldots}_{GŘ}) = -2 - s$

$q = 10^{-1}, \; a_1 = 5 \cdot 10^{-1}$

$|q| < 1$ pl.

$-2{,}\overline{5} = -2 - \frac{5}{9} = -\frac{23}{9}$

$s = \frac{a_1}{1-q} = \frac{5 \cdot 10^{-1}}{1-\frac{1}{10}} = \frac{\frac{5}{10}}{\frac{9}{10}} = \frac{50}{90} = \frac{5}{9}$

c) $-0{,}8\overline{4} = -(0{,}8 + 0{,}04 + 0{,}004 + 0{,}0004 + \ldots) = -(0{,}8 + \underbrace{4 \cdot 10^{-2} + 4 \cdot 10^{-3} + \ldots}_{GŘ}) = -(0{,}8 + s)$

$q = 10^{-1} = \frac{1}{10} \quad a_1 = 4 \cdot 10^{-2} = \frac{4}{100}$

$|q| < 1$ pl.

$-0{,}8\overline{4} = -\left(\frac{8}{10} + \frac{4}{90}\right) = -\frac{72+4}{90} = -\frac{76}{90}$

$-0{,}8\overline{4} = -\frac{38}{45}$

$s = \frac{a_1}{1-q} = \frac{\frac{4}{100}}{1-\frac{1}{10}} = \frac{\frac{4}{100}}{\frac{9}{10}} = \frac{40}{900} = \frac{4}{90}$

d) $0,\overline{12} = 0,121212\ldots = 12\cdot 10^{-2}+12\cdot 10^{-4}+12\cdot 10^{-6}+\ldots = 12(\underbrace{10^{-2}+10^{-4}+10^{-6}+\ldots}) =$

$= 12\cdot\frac{1}{99} = \frac{12}{99} = \frac{4}{33}$

[GŘ $q=10^{-2}=\frac{1}{100}$
$|q|<1$
podm. konvergence platí
$s=\frac{a_1}{1-q}=\frac{10^{-2}}{1-10^{-2}}=\frac{\frac{1}{100}}{1-\frac{1}{100}}=\frac{\frac{1}{100}}{\frac{99}{100}}=\frac{1}{99}$]

[nebo nevytýkat 12
$\Rightarrow a_1 = 12\cdot 10^{-2} \Rightarrow s=\frac{12\cdot 10^{-2}}{1-\frac{1}{100}}=\frac{12}{99}$]

e) $5,4\overline{87} = 5+\frac{4}{10}+87\cdot 10^{-3}+87\cdot 10^{-5}+\ldots =$

$=\frac{54}{10}+87(\underbrace{10^{-3}+10^{-5}+\ldots}) = \frac{54}{10}+\frac{87\cdot 1}{990} = \frac{54\cdot 99+87}{990}=\frac{5433}{990}=\frac{1811}{330}$

[GŘ: $q=10^{-2}$ $|q|<1$ podm. konv. platí
$s=\frac{a_1}{1-q}=\frac{10^{-3}}{1-10^{-2}}=\frac{\frac{1}{1000}}{1-\frac{1}{100}}=\frac{\frac{1}{1000}}{\frac{99}{100}}=\frac{100}{99\cdot 1000}=\frac{1}{990}$]

nebo
$5,4\overline{87}=\frac{54}{10}+\underbrace{87\cdot 10^{-3}+87\cdot 10^{-5}+87\cdot 10^{-7}+\ldots}=\frac{54}{10}+\frac{29}{33}=\frac{54\cdot 33+290}{330}=\frac{1811}{330}$

[GŘ: $q=10^{-2}$ $|q|<1$ podm. konv. platí
$s=\frac{a_1}{1-q}=\frac{87\cdot 10^{-3}}{1-10^{-2}}=\frac{\frac{87}{1000}}{1-\frac{1}{100}}=\frac{\frac{87}{1000}}{\frac{99}{100}}=\frac{87\cdot 100}{99\cdot 1000}=\frac{87}{99}=\frac{29}{33}$]

f) $-0,3\overline{25} = -(0,3+0,025+0,00025+0,0000025+\ldots)$
3Ř/143

$=-\left(\frac{3}{10}+\frac{25}{1000}+25\cdot 10^{-5}+25\cdot 10^{-7}+\ldots\right)$

$=-\left[\frac{3}{10}+25(\underbrace{10^{-3}+10^{-5}+10^{-7}+\ldots})\right)=-\left(\frac{3}{10}+25\cdot\frac{1}{990}\right)=$

[GŘ: $q=10^{-2}$ $|q|<1$ podm. konv. platí
$s=\frac{a_1}{1-q}=\frac{10^{-3}}{1-10^{-2}}=\frac{\frac{1}{1000}}{1-\frac{1}{100}}=\frac{\frac{1}{1000}}{\frac{99}{100}}=\frac{100}{99\cdot 1000}=\frac{1}{990}$]

$=-\frac{3\cdot 99+25}{990}=-\frac{297+25}{990}=-\frac{322}{990}=-\frac{161}{495}$

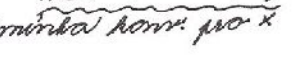
[v 1. ROČ. ČTYŘ. (5. ROČ. OSMIL)
$0,3\overline{25}=0,32525\ldots=\frac{3}{10}+0,02525\ldots$

$a=0,025|25\ldots$
$1000a=25,2525\ldots$
$10a=0,2525\ldots$ $\}\ominus$
$990a=25$
$a=\frac{25}{990}$]

(7) Zjistit, pro která $x\in R$ jsou řady konvergentní a určit jejich součet

3.28 a) $\sum_{n=1}^{\infty}\left(\frac{1}{x}\right)^n=\underbrace{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots}=x^{-1}+x^{-2}+x^{-3}+\ldots$ $x\neq 0$

GŘ: $q=\frac{\frac{1}{x^2}}{\frac{1}{x}}=\frac{x}{x^2}=\frac{1}{x}$ podm. konverg. $|q|<1$
$\left|\frac{1}{x}\right|<1$
$\frac{1}{|x|}<1$ $/\cdot|x|$, $|x|>0$
$1<|x|$
$|x|>1$, $x\neq 0$



$x\in(-\infty,-1)\cup(1,+\infty)$
podmínka konv. pro x

$s=\frac{a_1}{1-q}=\frac{\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{1}{x}}{\frac{x-1}{x}}=\frac{x}{x(x-1)}=\frac{1}{x-1}$

b) $\sum_{n=1}^{\infty}(1-5x)^n = (1-5x)^1+(1-5x)^2+(1-5x)^3+(1-5x)^4+\dots$

GŘ: $q=\frac{(1-5x)^2}{(1-5x)}=1-5x$ podm. konverg. $|q|<1$

$|1-5x|<1$

$|5x-1|<1 \quad |:5$!

$|x-\frac{1}{5}|<\frac{1}{5}$

$x_0=\frac{1}{5}$

$0 \quad \frac{1}{5} \quad \frac{2}{5}$

$s=\frac{a_1}{1-q}=\frac{1-5x}{1-(1-5x)}=\frac{1-5x}{5x} \quad \Leftarrow \quad x\in(0,\frac{2}{5})$

⑧ Řešte v R

3.29 a) $\sum_{n=1}^{\infty}(3x)^{n-1}=10$

$(3x)^0+(3x)^1+(3x)^2+\dots=10$

$\underbrace{1+3x+9x^2+\dots}=10$

GŘ: $q=\frac{9x^2}{3x}=\frac{3x}{1}=3x$ podm. konv. $|q|<1$

$|3x|<1$

$3|x|<1$

$|x|<\frac{1}{3}$

$-\frac{1}{3} \quad 0 \quad \frac{1}{3}$

$s=\frac{a_1}{1-q}=\frac{1}{1-3x} \quad \Leftarrow \quad x\in(-\frac{1}{3},\frac{1}{3})$

rovnice má tvar $\quad D=(-\frac{1}{3},\frac{1}{3})$

$\frac{1}{1-3x}=10$

$1=10(1-3x)$

$1=10-30x$

$30x=9$

$x=\frac{9}{30}=\frac{3}{10}\in D \quad K=\{\frac{3}{10}\}$

b) $\sum_{n=1}^{\infty}\left(\frac{2}{x}\right)^{n-1}=\frac{4x-3}{3x-4}$ $\quad D_0=R,\ D'=R-\{0,\frac{4}{3}\}$ $\quad [3x-4\neq0,\ x\neq\frac{4}{3}]$

$1+\frac{2}{x}+\left(\frac{2}{x}\right)^2+\left(\frac{2}{x}\right)^3+\dots=\frac{4x-3}{3x-4}$ $\quad [(\frac{2}{x})^{1-1}=(\frac{2}{x})^0=1$ pro $x\neq0]$

GŘ: $q=\frac{2}{x}$, $a_1=1$ → podm. konv. $|q|<1$ pl.

$|\frac{2}{x}|<1$

$s=\frac{a_1}{1-q}=\frac{1}{1-\frac{2}{x}}=\frac{1}{\frac{x-2}{x}}$ $\quad \frac{2}{|x|}<1 \quad |\cdot|x|_{>0}$

$s=\frac{x}{x-2}$ $\quad 2<|x|$

$|x|>2$

$-2 \quad 0 \quad 2$

-ce má tvar: $\frac{x}{x-2}=\frac{4x-3}{3x-4}$ $\quad x\in(-\infty,-2)\cup(2,+\infty)\cap D'$

$D=(-\infty,-2)\cup(2,+\infty)$

$x(3x-4)=(4x-3)(x-2)$

$3x^2-4x=4x^2-8x-3x+6$

$0=x^2-7x+6$

$0=(x-6)(x-1)$

$x_1=6\in D \quad x_2=1\notin D \quad K=\{6\}$

c) $\sum_{n=1}^{\infty} (x+2)^{2n} = \frac{1}{3}$

$$\underbrace{(x+2)^2 + (x+2)^4 + (x+2)^6 + \ldots}\ = \frac{1}{3}$$

GŘ: $a_1 = (x+2)^2 \quad q = \frac{(x+2)^4}{(x+2)^2} = (x+2)^2$ podm. konv. $|q| < 1$

$$|\underbrace{(x+2)^2}_{>0}| < 1$$

$$(x+2)^2 < 1$$

$$x^2 + 4x + 4 < 1 \quad \text{ANULOVAT}$$

$$x^2 + 4x + 3 < 0$$

$$(x+3)(x+1) < 0 \qquad x_0 = -3 \quad x_0 = -1$$

$$x \in (-3, -1)$$

$$s = \frac{a_1}{1-q}$$

$$s = \frac{(x+2)^2}{1-(x+2)^2} = \frac{(x+2)^2}{[1-(x+2)][1+(x+2)]}$$

$$s = \frac{(x+2)^2}{(-1-x)(3+x)} = \frac{(x+2)^2}{-(1+x)(x+3)}$$

$$\mathcal{D} = (-3, -1)$$

rce má tvar:

$$\frac{(x+2)^2}{-(1+x)(x+3)} = \frac{1}{3}$$

$$3(x+2)^2 = -(x+1)(x+3)$$

$$3(x^2+4x+4) = -(x^2+3x+x+3)$$

$$3x^2+12x+12 = -x^2-4x-3$$

$$4x^2+16x+15 = 0$$

$$(a=4,\ b=16,\ c=15)$$

$$x_{1,2} = \frac{-16 \pm \sqrt{16^2 - 4.4.15}}{8}$$

$$x_{1,2} = \frac{-16 \pm \sqrt{256-240}}{8} = \frac{-16 \pm 4}{8}$$

$$x_{1,2} = \begin{cases} -\frac{12}{8} = -\frac{3}{2} = -1{,}5 \in \mathcal{D} \\ -\frac{20}{8} = -\frac{5}{2} = -2{,}5 \in \mathcal{D} \end{cases}$$

$$\mathcal{K} = \left\{-\frac{3}{2}, -\frac{5}{2}\right\}$$

d) $\sum_{n=1}^{\infty} \frac{1}{x^{2n}} = \frac{1}{x^2-1}$

$$\underbrace{\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots}\ = \frac{1}{x^2-1}$$

GŘ: $a_1 = \frac{1}{x^2}, \quad q = \frac{1}{x^2}$ podm. konvergence $|q| < 1$

$$\left|\frac{1}{x^2}\right| < 1$$

$$\frac{1}{|x^2|} < 1 \qquad \left[\underbrace{x^2 \geq 0 \text{ pro } \forall x \in \mathbb{R} \wedge x \neq 0}\ \ x^2 > 0 \Rightarrow |x^2| = x^2\right]$$

$$\frac{1}{x^2} < 1 \quad / \cdot \underset{>0}{x^2}$$

$$1 < x^2$$

$$|x| > 1$$

$$s = \frac{a_1}{1-q} = \frac{\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{\frac{1}{x^2}}{\frac{x^2-1}{x^2}} = \frac{x^2}{x^2(x^2-1)}$$

$$s = \frac{1}{x^2-1}$$

$$\mathcal{D} = (-\infty, -1) \cup (1, +\infty)$$

- rce má tvar: $s = \frac{1}{x^2-1}$

$$\frac{1}{x^2-1} = \frac{1}{x^2-1}$$

$$0x = 0$$

$$\Rightarrow \mathcal{K} = \mathcal{D} = (-\infty, -1) \cup (1, +\infty)$$